arride learning

Function

Contents

Syllabus

Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial rational, trigonometric, exponential and logarithmic functions.

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FUNCTION

Definition :

Let A and B be two given non empty sets and if each element $a \in A$ is associated with a unique element $b \in B$ under a rule f, then this relation is called **function**.

Here b, is called the image of a and a is called the pre- image of b under f.

Note : (i) Every element of A should be associated with B but vice-versa is not essential.

(ii) Every element of A should be associated with a unique (one and only one) element of but any element of B can have two or more relations in A.

1. Representation of Function :

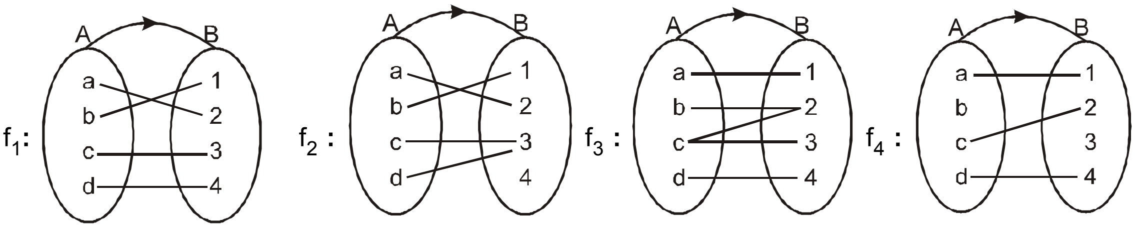
It can be done by three methods :

(A) By Mapping

(B) By Algebraic Method

(C) In the form of Ordered pairs

(A) Mapping : It shows the graphical aspect of the relation of the elements of A with the elements of B .



In the above given mappings rule f_1 and f_2 shows a function because each element of A is associated with a unique element of B. Whereas f_3 and f_4 are not function because in f_3, element c is associated with two elements of B, and in f_4 , b is not associated with any element of B, which do not follow the definition of function. In f_2, c and d are associated with same element, still it obeys the rule of definition of function because it does not tell that every element of A should be associated with different elements of B.

(B) Algebraic Method :

It shows the relation between the elements of two sets in the form of two variables x and y where x is independent variable and y is dependent variable.

If A and B be two given sets

$A = \{ 1,2,3 \}$, $B = \{5,7,9\}$ then $f : A \to B$, $y = f(x) = 2x + 3$.

(C) In the form of ordered pairs :

A function $f : A \to B$ can be expressed as a set of ordered pairs in which first element of every ordered pair is a member of A and second element is the member of B. So f is a set of ordered pairs (a, b) such that :

(i) a is an element of A

(ii) b is an element of B

(iii) Two ordered pairs should not have the same first element.

Example 1.

If A and B are two sets such that $A = \{1, 2, 3\}$ and $B = \{5,7,9\}$ and if a function is defined from $f : A \to B$, $f(x) = 2x + 3$ then find a function in the form of ordered pairs.

Sol. $f : A \to B$, $f(x) = 2x + 3$

$f(1) = 2.1 + 3 = 5$, $f(2) = 2.2 + 3 = 7$

$f(3) = 2.3 + 3 = 9$ $\therefore$ $f : \{(1,5) ; (2,7) ,(3,9) \}$

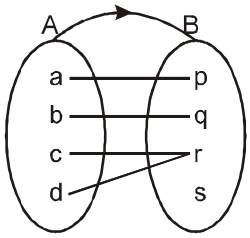
Domain, Co-domain and Range :

If a function f is defined from a set of A to set B then for f: A $\rightarrow$ B set A is called the **domain** of function f and set B is called the **co-domain** of function f. The set of the f- images of the elements of A is called the **range** of function f.

In other words, we can say

Domain = All possible values of x for which f(x) exists.

Range = For all values of x, all possible values of f(x).



Domain = {a,b,c,d} = A

Co-domain = {p,q,r,s} = B

Range = {p,q,r}

Algebra of functions :

Let f and g be two given functions and their domain are D_f and D_g respectively, then the sum, difference, product and quotient functions are defined as :

(a) $(f + g)(x) = f(x) + g(x),\ \forall\ x \in D_f \cap D_g$

(b) $(f - g)(x) = f(x) - g(x),\ \forall\ x \in D_f \cap D_g$

(c) $(f\ .\ g)(x) = f(x)\ .\ g(x),\ \forall\ x \in D_f \cap D_g$

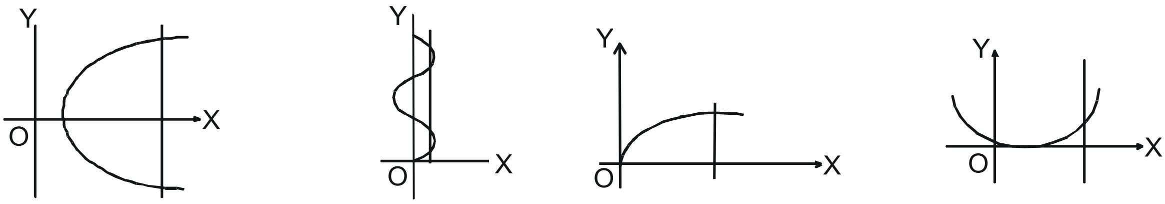
(d) $(f/g)(x) = \dfrac{f(x)}{g(x)}$; $g(x) \neq 0,\ \forall\ x \in D_f \cap D_g$

Testing for a function :

A relation f : A $\rightarrow$ B is a function or not, it can be checked by following methods.

(a) See Article 1 (A)

(b) Vertical Line Test : If we are given a graph of the relation then we can check whether the given relation is function or not . If it is possible to draw a vertical line which cuts the given curve at more than one point then given relation is not a function and when this vertical line means line parallel to Y - axis cuts the curve at only one point then it is a function.



(i) (ii) (iii) (iv)

fig. (iii) and (iv) represents a function.

Value of the Function :

If y = f(x) is any function defined in R, then for any given value of x (say x = a), the value of the function f(x) can be obtained by substituting x = a in it and it is denoted by f(a).

IMPORTANT TYPES OF FUNCTION :

(i) POLYNOMIAL FUNCTION :

If a function if is called by $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n$ where n is a non negative integer and $a_0, a_1, a_2, ...a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note : (a) A polynomial of degree one with no constant term is called an odd linear function. i.e. $f(x) = ax, a \neq 0$.

(b) There are two polynomial functions, satisfying the relation; $f(x).\ f(1/x) = f(x) + f(1/x)$.

They are (i) $f(x) = x^n + 1$ & (ii) $f(x) = 1 - x^n$, where n is a positive integer.

(ii) ALGEBRAIC FUNCTION :

y is an algebraic function of x, if it is a function that satisfies an algebraic equation of the form P_0

(x) $y^n + P_1(x)y^{n-1} + ... + P_{n-1}(x)y + P_n(x) = 0$ where n is a positive integer and $P_0(x), P_1(x)$.... are Polynomials in x. e.g. y = |x| is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$. Note that all polynomial functions are Algebraic but not the converse. A functions that is not algebraic is called **TRANSCENDENTAL** function.

(iii) FRACTIONAL RATIONAL FUNCTION :

A rational function is a function of the form. $y = f(x) = \frac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials & $h(x) \neq 0$

(iv) EXPONENTIAL FUNCTION :

A function $f(x) = a^x = e^{x/na}$ $(a > 0)$, $a \neq 1$, $x \in R$) is called an exponential function.
The inverse of the exponential function is called the logarithmic function, i.e. $g(x) = \log_a x$. Note that f(x) & g (x) are inverse of each other & their graphs are as shown.

(v) ABSOLUTE VALUE FUNCTION :

A function y = f (x) = |x| is called the absolute value function or Modulus function.

It is defined as : $y = |x| - \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

(VI) SIGNUM FUNCTION :

A function y = sgn (x) is defined as follows

$$y = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0 \end{cases}$$

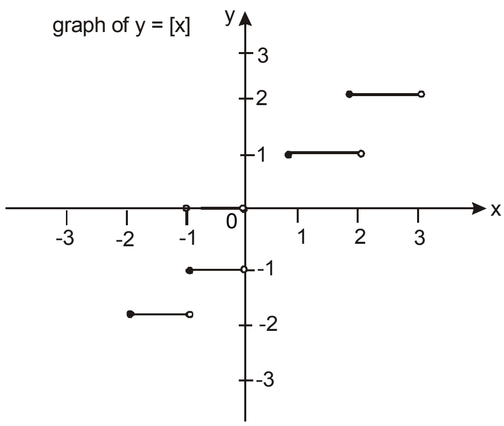
It is also written as $\text{sgn}(x) = \begin{cases} \frac{|x|}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

$x \neq 0$; $f(0) = 0$

(vii) GREATEST INTEGER OR STEP UP FUNCTION :

The function y = f (x) = [x] is called the greatest integer function where [x] denotes the greatest integer less than or equal to x. Note that for :

-2	$-2 \leq x < -1$
-1	$-1 \leq x < 0$
0	$0 \leq x < 1$
1	$1 \leq x\ 2$
2	$2 \leq x < 3$



PROPERTIES OF GREATEST INTEGER FUNCTION :

(a) $[x] \leq x < [x] + 1$ and

$x - 1 < [x] \leq x$, $0 \leq x - [x] < 1$

(b) $[x + m] = [x] + m$ if m is an integer.

(c) $[x] + [y] \leq [x + y] \leq [x] + [y] + 1$

(d) $[x] + [-x] = 0$ if x is an integer $= -1$ otherwise

(e) In general $[mx] \neq m[x]$

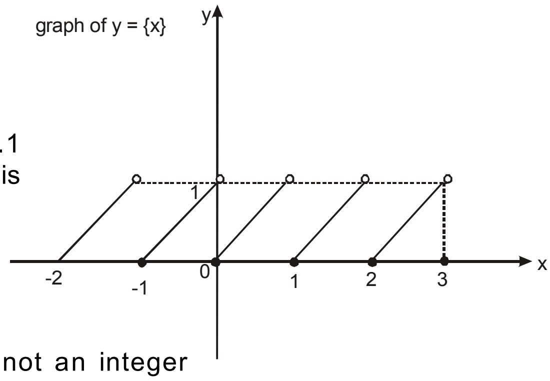
(viii) FRACTIONAL PART FUNCTION :

It is defined as :

g (x) = {x} = x – [x]

e.g. the fractional part of the no. 2.1 is 2.1 –2 = 0.1 and the fractional part of –3. 7 is 0.3 The period of this function is 1

and graph of this function is as shown.



Properties:

(a) {x + m} = {x} if m is an integer

(b) {x} + {–x} = 0, if x is an integer and 1 if x is not an integer

DOMAINS AND RANGES OF COMMON FUNCTION

	Function	**Domain**	**Range**
A. ALGEBRAIC FUNCTIONS :			
(i)	x^n, $(n \in N)$	R = (set of real numbers)	R, if n is odd $R^+ \cup \{0\}$, if n is even
(ii)	$\frac{1}{x^n}$, $(n \in N)$	R – {0}	R – {0}, if n is odd R^+, if n is even
(iii)	$x^{1/n}$, $(n \in N)$	R, if n is odd $R^+ \cup \{0\}$, if n is even	R, if n is odd $R^+ \cup \{0\}$, if n is even
(iv)	$\frac{1}{x^{1/n}}$, $(n \in N)$	R – {0}, if n is odd R^+, if n is even	R – {0}, if n is odd R^+, if n is even
B. TRIGONOMETRIC FUNCTIONS :			
(i)	sin x	R	[–1, 1]
(ii)	cos x	R	[–1, 1]
(iii)	tan x	$R - (2k+1)\frac{\pi}{2}$, $k \in I$	R
(iv)	sec x	$R - (2k+1)\frac{\pi}{2}$, $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(v)	cosec x	$R - k\pi$, $k \in I$	$(-\infty, -1] \cup [1, \infty)$
(vi)	cot x	$R - k\pi$, $k \in I$	R
C. INVERSE TRIGONOMETRIC FUNCTION :			
(i)	$\sin^{-1}x$	[–1,1]	$\left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$
(ii)	$\cos^{-1}x$	[–1, +1]	$[0, \pi]$
(iii)	$\tan^{-1}x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(iv)	$\text{cosec}^{-1}x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
(v)	$\cot^{-1}x$	R	$(0, \pi)$
D. EXPONENTIAL FUNCTIONS :			
(i)	e^x	R	R^+
(ii)	$e^{1/x}$	R – {0}	$R^+ - \{1\}$
(iii)	a^x, a > 0	R	R^+
(iv)	$a^{1/x}$, a > 0	R – {0}	$R^+ - \{1\}$

E. LOGARITHMIC FUNCTIONS :

(i)	$\log_a x$, $(a > 0)$ $(a \neq 1)$	R^+	R
(ii)	$\log_x a = \dfrac{1}{\log_a x}$ $(a > 0)$ $(a \neq 1)$	$R^+ - \{1\}$	$R - \{0\}$

F. INTEGRAL PART FUNCTIONS :

(i)	$[x]$	R	I
(ii)	$\dfrac{1}{[x]}$	$R - [0, 1)$	$\dfrac{1}{n}, n \in I - \{0\}$

G. FRACTION PART FUNCTIONS :

(i)	$\{x\}$	R	$[0, 1)$
(ii)	$\dfrac{1}{\{x\}}$	$R - I$	$(1, \infty)$

H. MODULUS FUNCTIONS :

(i)	$\mid x \mid$	R	$R^+ \cup \{0\}$
(ii)	$\dfrac{1}{\mid x \mid}$	$R - \{0\}$	R^+

I. SIGNUM FUNCTION :

$\text{sgn}(x) = \dfrac{\mid x \mid}{x}, x \neq 0$ $= 0,$ x $x = 0$	R	$\{-1, 0, 1\}$

J. CONSTANT FUNCTION :

say $f(x) = c$	R	$\{c\}$

EQUAL OR IDENTICAL FUNCTION :

Two function f & g are said to be equal if :

(i) The domain of f = the domain of g

(ii) The range of f = range of g and

(iii) $f(x) = g(x)$, for every x belonging to their common domain.

eg. $f(x) = \dfrac{1}{x}$ & $g(x) = \dfrac{x}{x^2}$ are identical functions.

CLASSIFICATION OF FUNCTIONS :

One – One function (Injective mapping) :

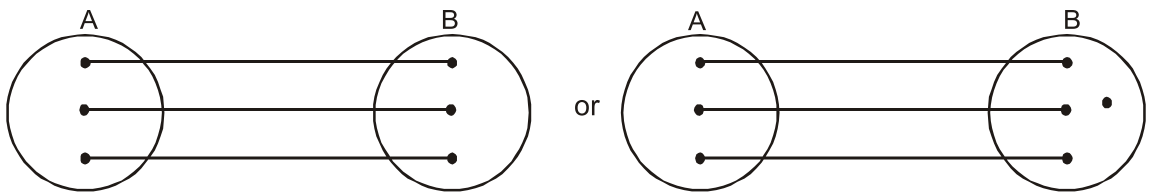
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B. Thus for $x_1, x_2 \in A$ & $f(x_1)$.

$f(x_2) \in B$, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$ or $x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$.

Note: (i) Any continuous function which is entirely increasing or decreasing in whole domain, then f(x) is one-one.

(ii) If a function in one-one, any line parallel to x-axis cuts the graph of the function at atmost one point

Diagramatically an injective mapping can be shown

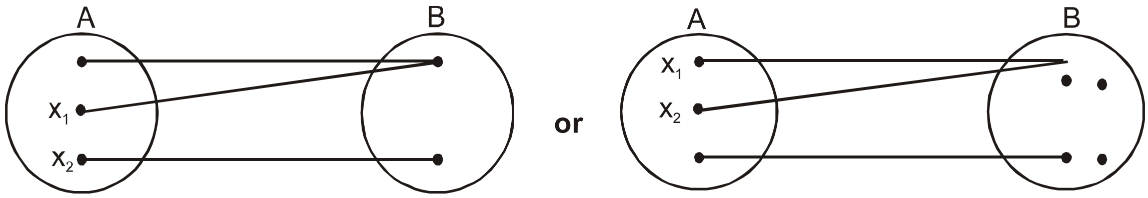


Many-one function :

A function f: A → B is said to be a many one function if two or more elements of A have the same f image in B.

Thus f : A → B is many one if for ; $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ but $x_1 \neq x_{20}$

Diagramatically an many one mapping can be shown



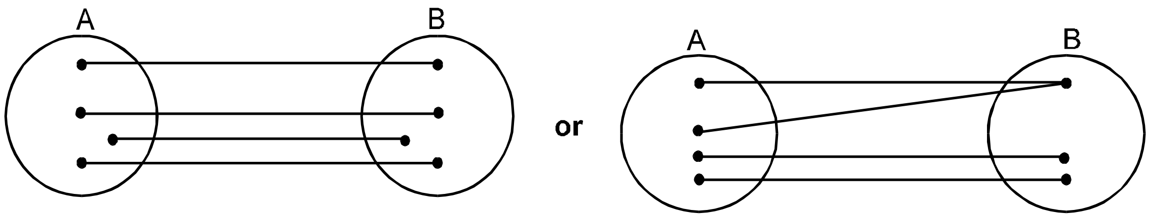
Note : (i) Any continuous function which has atleast one local maximum or local minimum, then f(x) is many-one.

(ii) If a function is many - one, atleast one line parallel to x-axis will intersect the graph of function atleast twice

Onto function (Surjective mapping) :

If the function f : A → B is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus f : A → B is surjective if $\forall$ b $\in$ B, $\exists$ some a $\in$ A such that f(a) = b

Diagramatically surjective mapping can be shown

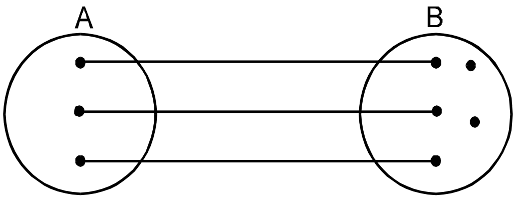


Note that : If range = co-domain, then f (x) is onto .

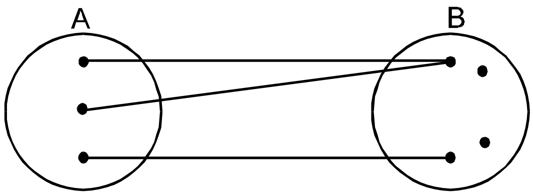
Into function :

If f : A → B is such that there exists atleast one element in co-domain which is not the image of any element in domain, then f (x) is into.

Diagramatically into function can be shown



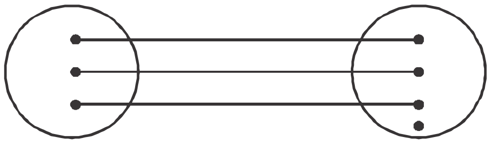
or



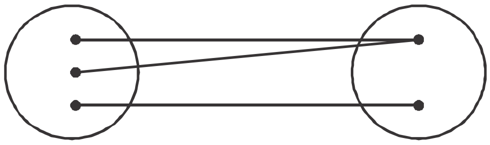
Thus a function can be one of these four types :

(a) one-one onto (injective & surjective)

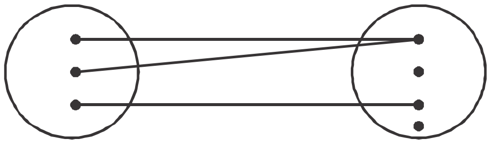
(b) one-one into (injective but not surjective)



(c) many-one onto (surjective but not injective)



(d) many-one into (neither surjective nor injective)



Note : (i) If f is both injective & surjective, then it is called a **Bijective** mapping. The bijective functions are also named as invertible, non singular or biuniform functions.

(ii) If a set A contains n distinct elements then the number of different functions defined from $A \to A$ is n^n & out of it n! are one one.

(iii) If f and g both are onto, then gof or fog may or may not be onto.

(iv) The composite of two bijections is a bijection iff f and g are two bijections such that gof is defined, then gof is also a bijection only when co-domain of f is equal to the domain of g.

IDENTITY FUNCTION :

The function $f : A \to A$ defined by $f(x) = x \ \forall\ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function is a bijection.

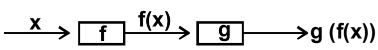
CONSTANT FUNCTION :

Thus $f : A \to B$ is said to be constant function if every element of A has the same f image in B.

Thus $f : A \to B$; $f(x) = c,\ \forall\ x \in A,\ c \in B$ is constant function. Note that the range of a constant function is a singleton and a constant function may be one-one or many-one, onto or into.

COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTION:

Let $f : A \to B$ & $g : B \to C$ be two functions. Then the function gof : $A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.

Diagrammatically 

Thus the image of every $x \in A$ under the function gof is the g-image of f-image of x.

Note that gof is defined only if $\forall\ x \in A$, f (x) is an element of the domain of g so that we can take its g-image. Hence for the product gof two functions f & g, **the range of f must be a subset of the domain of g.**

Properties of composite functions:

(i) In general composite of functions is not commutative i.e. gof $\neq$ fog.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that fo (goh) & (fog) oh are defined, then fo (goh) = (fog) oh.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

IMPLICIT & EXPLICIT FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **implicit function**. For e.g. the equating $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit function.**

INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x,\ \forall\ x \in A\ \&\ y \in B$. $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{ (f(x), x)) \mid (x, f(x)) \in f\}$

PROPERTIES OF INVERSE FUNCTION :

(i) The inverse of a bijection in unique.

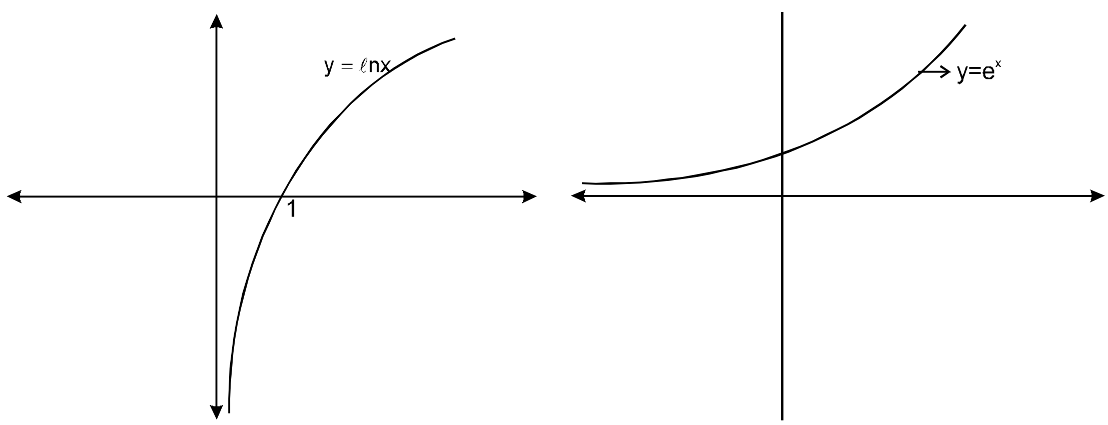
(ii) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then fog = I_B and gof = I_A, where I_A & I_B are identity functions on the sets A & B respectively. If fof = I, then f is inverse of itself Note that the graph of f & g are the mirror images of each other in the line y = x. As shown in the figure given below a point (x', y') corresponding to $y = \ell nx\ (x > 0)$ changes to (y', x') corresponding to $y = e^x$, the changed form of $x = e^y$.

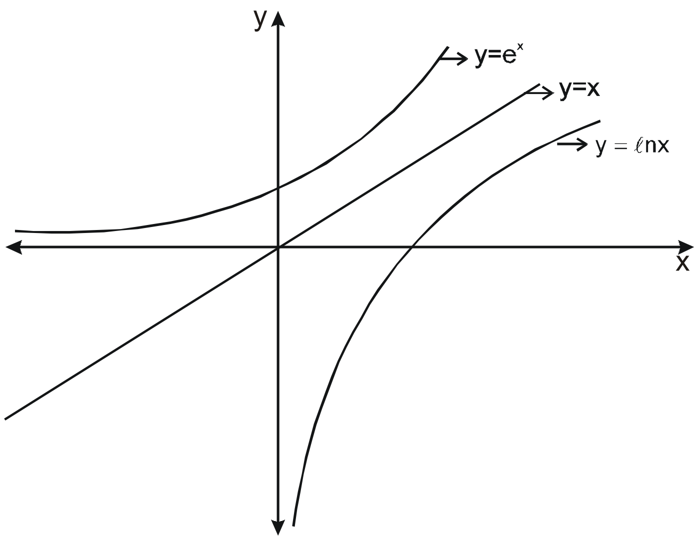
(iii) The inverse of a bijection is also a bijection.

(iv) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1} o\ g^{-1}$.

(v) If f(x) and g(x) are inverse function of each other, then $f'(g(x)) = \frac{1}{g'(x)}$

ODD & EVEN FUNCTIONS :





If a function is such that whenever 'x' is in it's domain '–x' is also in it's domain & it satisfies.

f (x) – f (–x) = 0 it is an even function

f (x) + f (–x) = 0 it is an odd function

Note: (a) f (x) – f (–x) = 0 ⇒ f(x) is even & f(x) + f (– x) = 0 ⇒ f (x) is odd.

(b) A function may neither be odd nor even.

(c) Inverse of an even function is not defined.

(d) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(e) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

$$\text{e.g. } f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{ODD}}$$

(f) The only function which is defined on the entire number line & even and odd at the same time is f(x) = 0

(g) If f(x) and g(x) both are even or both are odd then the function f(x). g(x). will be even but if any one of them is odd then f.g will be odd.

PERIODIC FUNCTION :

A function f(x) is called periodic if there exists a positive number T (T >0) called the period of the function such that f(x + T) = f(x), for all values of x within the domain of x.

e..g The function sinx & cosx both are periodic over 2π & tan x is periodic over π.

Note: (a) f(T) = f (0) = f(–T) , where 'T' is the period.

(b) Inverse of a periodic function does not exist.

(c) Every constant function is always periodic, with no fundamental period.

(d) If f(x) has a period T & g (x) also has a period T then it does not mean that f(x) + g (x) must have a period T. e.g. f (x) = | sin x | + | cos x |.

(e) If f(x) has period p, then $\frac{1}{f(x)}$ and $\sqrt{f(x)}$ also has a period p.

(f) If f(x) has period T then f(ax + b) has a period T/|a| (a > 0).

GENERAL :

If x, y are independent variables, then :

(i) $f(xy) = f(x) + f(y) \Rightarrow f(x) = k \ \ell n \ x$ or $f(x) = 0$

(ii) $f(xy) = f(x) \, . \, f(y) \Rightarrow f(x)^n = x^n, \ n \in R$

(iii) $f(x + y) = f(x) \, . \, f(y) \Rightarrow f(x) = a^{kx}$

(iv) $f(x + y) = f(x) + f(y) \Rightarrow f(x) = kx$, where k is a constant.

(v) There are only two polynomial functions, satisfying the relation ; $f(x). f(1/x) = f(x) + f(1/x)$, which are $f(x) = 1 \pm x^n$

EXERCISE # 1

PART - I : OBJECTIVE QUESTIONS

*** Marked Questions are having more than one correct option.***

Section (A) : Definition of function and Domain and range

A-1. If x, y $\in$ R, then which of the following rules is not a function-

(A) $y = 9 - x^2$ (B) $y = 2x^2$ (C) $y = \sqrt{x} - |x|$ (D) $y = x^2 + 1$

A-2. If $f(x) = \cos\left[\frac{\pi^2}{2}\right]x + \sin\left[-\frac{\pi^2}{2}\right]x$, where [x] denotes the greatest integer function, then which of the following is not correct –

(A) f(0) = 1 (B) $f\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}+1}$ (C) $f\left(\frac{\pi}{2}\right) = 0$ (D) $f(\pi) = 0$

A-3. If $f(x) = \frac{2x}{1-x^2}$, then f (tan θ) equals-

(A) cot 2θ (B) tan 2 θ (C) sec 2θ (D) cos 2 θ

A-4. If $f(x) = \frac{x}{x+1}$, then $\frac{f(a/b)}{f(b/a)} =$

(A) ab (B) a/b (C) b/a (D) 1

A-5. If f(x + ay, x – ay) = axy, then f (x,y) equals-

(A) $\frac{x^2+y^2}{4}$ (B) $\frac{x^2-y^2}{4}$ (C) x^2 (D) y^2

A-6. If $f(x) = \frac{ax-c}{cx-a} = y$, then f(y) equals –

(A) x (B) $\frac{1}{x}$ (C) 1 (D) 0

A-7. The domain of the function $f(x) = \frac{\sqrt{-\log_{0.3}(x-1)}}{\sqrt{x^2+2x+8}}$ is

(A) (1, 4) (B) (– 2, 4) (C) (2, 4) (D) [2, ∞)

A-8. The domain of definition of $f(x) = \sin^{-1}(|x-1| - 2)$ is:

(A) [– 2, 0] $\cup$ [2, 4] (B) (– 2, 0) $\cup$ (2, 4) (C) [– 2, 0] $\cup$ [1, 3] (D) (– 2, 0) $\cup$ (1, 3)

A-9. The function $f(x) = \cot^{-1}\sqrt{(x+3)x} + \cos^{-1}\sqrt{x^2+3x+1}$ is defined on the set S, where S is equal to:

(A) {0, 3} (B) (0, 3) (C) {0, – 3} (D) [– 3, 0]

A-10. Range of $f(x) = 4^x + 2^x + 1$ is
(A) $(0, \infty)$ (B) $(1, \infty)$ (C) $(2, \infty)$ (D) $(3, \infty)$

A-11. Range of $f(x) = \log(3x^2 - 4x + 5)$ is

(A) $\left[\log \frac{11}{3}, \infty\right)$ (B) $[\log 10, \infty)$ (C) $\left[\log \frac{11}{6}, \infty\right)$ (D) $\left[\log \frac{11}{12}, \infty\right)$

A-12. Range of $f(x) = \log_{\sqrt{5}} \{\sqrt{2}(\sin x - \cos x) + 3\}$ is

(A) $[0, 1]$ (B) $[0, 2]$ (C) $\left[0, \frac{3}{2}\right]$ (D) none of these

A-13. Let f(x) be a function whose domain is $[-5, 7]$. Let $g(x) = |2x + 5|$. Then domain of (fog) (x) is
(A) $[-4, 1]$ (B) $[-5, 1]$ (C) $[-6, 1]$ (D) none of these

Section (B) : Composite function and Classification of function

B-1. If $f(x) = \frac{ax+b}{cx+d}$, then (fof) (x) = x, provided that
(A) $d + a = 0$ (B) $d - a = 0$ (C) $a = b = c = d = 1$ (D) $a = b = 1$

B-2. The function $f : [2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one–one & onto if
(A) $Y = R$ (B) $Y = [1, \infty)$ (C) $Y = [4, \infty)$ (D) $Y = [5, \infty)$

B-3. Let $f : R \to R$ be a function defined by $f(x) = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$ then f is:

(A) one – one but not onto (B) onto but not one – one
(C) onto as well as one – one (D) neither onto nor one – one

B-4. Let $f : R \to R$ be a function defined by $f(x) = x^3 + x^2 + 3x + \sin x$. Then f is:
(A) one – one & onto (B) one – one & into
(C) many one & onto (D) many one & into

B-5*. Let $f(x) = \left(\frac{1-x}{1+x}\right)$, $0 \le x \le 1$ and $g(x) = 4x(1-x)$, $0 \le x \le 1$. then

(A) fog $= \frac{1-4x+4x^2}{1+4x-4x^2}$, $0 \le x \le 1$ (B) fog $= \frac{1-4x-4x^2}{1+4x-4x^2}$, $\frac{1}{2} \le x \le 1$

(C) gof $= \frac{8x(1-x)}{(1+x)^2}$, $0 \le x \le 1$ (D) gof $= \frac{8x(1+x)}{(1+x)^2}$, $0 \le x \le 1$

B-6. If 'f' and 'g' are bijective functions and gof is defined then gof must be:
(A) injective (B) surjective (C) bijective (D) into only

B-7*. $D \equiv [-1, 1]$ is the domain of the following functions, state which of them are injective.
(A) $f(x) = x^2$ (B) $g(x) = x^3$ (C) $h(x) = \sin 2x$ (D) $k(x) = \sin(\pi x/2)$

B-8*. Let $f : I \to R$ (where I is the set of positive integers) be a function defined by

$f(x) = \sqrt{x}$, then f is:

(A) one – one (B) many one (C) onto (D) into

Section (C) : Even/Odd function and Periodic function

C-1. The function $f(x) = \log\left(\frac{1+\sin x}{1-\sin x}\right)$ is

(A) even (B) odd
(C) neither even nor odd (D) both even & odd

C-2. The function $f(x) = [x] + \frac{1}{2}$, $x \notin I$ is a/an

(A) Even (B) odd (C) neither even nor odd (D) none of these

C-3. Fundamental period of f(x) = sec (sin x) is

(A) $\frac{\pi}{2}$ (B) 2π (C) π (D) aperiodic

C-4. If $f(x) = \sin\sqrt{[a]}\,x$ (where [.] denotes the greatest integer function) has π as its fundamental period, then:
(A) a = 1 (B) a = 9 (C) $a \in [1, 2)$ (D*) $a \in [4, 5)$

C-5. The fundamental period of the function,
$f(x) = x + a - [x + b] + \sin \pi x + \cos 2\pi x + \sin 3\pi x + \cos 4\pi x + \ldots\ldots + \sin (2n - 1)\pi x + \cos 2n\pi x$
for every $a, b \in R$ is: (where [ ] denotes the greatest integer function)
(A) 2 (B) 4 (C) 1 (D) 0

C-6*. The period of the function $f(x) = \sin^4 3x + \cos^4 3x$ is:
(A) $\pi/6$ (B) $\pi/3$ (C) $\pi/2$ (D) $\pi/12$

C-7*. The functions which are aperiodic are (where [x] denotes greatest integer function)
(A) y = [x + 1] (B) $y = \sin x^2$ (C) $y = \sin^2 x$ (D) $y = \sin^{-1} x$

C-8*. If $f: R \to [-1, 1]$, where $f(x) = \sin\frac{\pi}{2}[x]$, (where [ ] denotes the greatest integer function) then:

(A) f (x) is onto (B) f (x) is into (C) f (x) is periodic (D) f (x) is many one

C-9*. Identify the statement(s) which is/are incorrect ?
(A) the function $f(x) = \cos(\cos^{-1} x)$ is neither odd nor even
(B) the fundamental period of $f(x) = \cos(\sin x) + \cos(\cos x)$ is π
(C) the range of the function $f(x) = \cos(3 \sin x)$ is $[-1, 1]$
(D) none of these

Section (D) : Inverse of a function and functional Equations

D-1. The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is given by -

(A) $\log\left(\frac{x-2}{x-1}\right)^{1/2}$ (B) $\log\left(\frac{x-1}{x+1}\right)^{1/2}$ (C) $\log\left(\frac{x}{2-x}\right)^{1/2}$ (D) $\log\left(\frac{x-1}{3-x}\right)^{1/2}$

D-2. If $f(x) = \cos(\ell n\, x)$ then $f(x)\,f(y) - \frac{1}{2}\left[f\left(\frac{x}{y}\right) + f(xy)\right]$ has the value

(A) –1 (B) $\frac{1}{2}$ (C) –2 (D) None of these

D-3. If y = f(x) satisfies the condition $f\left(x+\frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ $(x \neq 0)$ then f(x) =

(A) $-x^2+2$ (B) $-x^2-2$
(C) $x^2-2,\ x \in R-\{0\}$ (D) $x^2-2,\ |x| \in [2, \infty)$

D-4. If f(1) = 1 and f(n + 1) = 2f(n) + 1 if $n \geq 1$, then f(n) is equal to
(A) $2^n + 1$ (B) 2^n (C) $2^n - 1$ (D) $2^{n-1} - 1$

PART - II : MISCELLANEOUS OBJECTIVE QUESTIONS

COMPREHENSIONS :
Read the following passage carefully and answer the questions.

Comprehension # 1

Let $f(x) = \frac{x^3}{3} + \frac{x^2}{2} + ax + b \quad \forall\ x \in R$

1. Least value of 'a' for which f(x) is injective function, is

(A) $\frac{1}{4}$ (B) 1 (C) $\frac{1}{2}$ (D) $\frac{1}{8}$

2. If a = – 1 then f(x) is
(A) bijective (B) many–one and onto (C) one–one and into (D) many– one and into

3. f(x) is invertible iff

(A) $a \in \left[\frac{1}{4}, \infty\right),\ b \in R$ (B) $a \in \left[\frac{1}{8}, \infty\right),\ b \in R$

(C) $a \in \left(-\infty, \frac{1}{4}\right],\ b \in R$ (D) $a \in \left(-\infty, \frac{1}{4}\right),\ b \in R$

Comprehension # 2

If f : [0, 2] → [0, 2] is a bijective function defined by $f(x) = ax^2 + bx + c$, where a, b, c, are non zero real numbers, then

4. f(2) =
(A) 2 (B) α where $a \in (0, 2)$ (C) 0 (D) cannot be determined

5. Which of the following is one of the roots f(x) = 0 is

(A) $\frac{1}{a}$ (B) $\frac{1}{b}$ (C) $\frac{1}{c}$ (D) $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

6. Which of the following is not a value of a ?

(A) $-\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $-\frac{1}{2}$ (D) 1

7. **Match the Column :**

Column-I		Column-II	
(A)	$f : R \to R,\ f(x) = \frac{x-1}{x^2 - 2x + 3}$	(p)	Injective but not surjective
(B)	$f : R - \{1\} \to R,\ f(x) = \frac{3x+2}{x-1}$	(q)	Surjective but not Injective
(C)	$f : [1, \infty) \to [1, \infty),\ f(x) = x^2 - 2x + 2$	(r)	Bijective
(D)	$f : R \to [0, \infty),\ f(x) = x^2$	(s)	Neither injective nor surjective

8. **Match the Column :**

Column-I		Column-II	
(A)	$f(x) = \tan x$	(p)	Even function
(B)	$f(x) = \frac{e^{2x}+1}{e^{2x}-1}$	(q)	Odd function
(C)	$f(x) = 0$	(r)	Neither even nor odd
(D)	$f(x) = 2^{-\lvert x(x-1) \rvert}$	(s)	Bounded function

ASSERTION-REASON TYPE

DIRECTION :

Each question contains Statement–1 (Assertion) and Statement–2 (Reason). For the following questions 4 answers (A), (B), (C) and (D) are given below, of which only one is correct.

(A) Statement-1 is True, Statement-2 is True; Statement-2 is a correct explanation for Statement-1.
(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1
(C) Statement-1 is True, Statement-2 is False.
(D) Statement-1 is False, Statement-2 is True.

9. **Statement - 1 :** $f(x) = \ln(\tan x)$ has range R.
Statement - 2 : $\ln(g(x))$ assumes all real values if $g(x)$ assumes all positive values.

10. **Statement - 1** If f (x) & g (x) both are one one, then f (g (x)) is also one one.
Statement - 2 If, $f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2$, then f(x) is one-one.

11. **Statement - 1** If y = f (x) is increasing in $[\alpha, \beta]$ then its range is $[f(\alpha), f(\beta)]$
Statement - 2 Every increasing function need not to be continuous.

12. **Statement - 1** Let $f : [0, 3] \to [1, 13]$ is defined by $f(x) = x^2 + x + 1$ then inverse is $f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}$
Statement - 2 **Many-one** function is not invertible

EXERCISE # 2

PART - I : OBJECTIVE QUESTIONS

Single choice

1. Domain of the function $f(x) = \dfrac{1}{\sqrt{\log_{\frac{1}{2}}(x^2 - 7x + 13)}}$ is :

(A) $x \in R$ (B) $x \in [3, 4]$ (C) $(3, 4)$ (D) $(-\infty, 3) \cup (4, \infty)$

2. Domain of the function $f(x) = \dfrac{1}{\sqrt{^{10}C_{x-1} - 3\,^{10}C_x}}$ contains the points :

(A) 9, 10, 11 (B) 9, 10, 12 (C) 9, 10 (D) all natural numbers

3. Domain of the function. $f(x) = \ln(x^2 - 2x + 3) + \dfrac{1}{\sqrt{x-2}}$ is :

(A) R (B) $[2, \infty)$ (C) $(2, \infty)$ (D) $(-\infty, 2) \cup (2, \infty)$

4. The domain of the function $f(x) = \sqrt{\dfrac{4 - x^2}{[x] + 2}}$, where [x] denotes the greatest integer less than or equal to x, is :

(A) $[-1, 2]$ (B) $(-\infty, -2)$
(C) $(-\infty, -2) \cup [-1, 2]$ (D) none of these

5. The image of the interval R when the mapping $f: R \to R$ given by $f(x) = \cot^{-1}(x^2 - 4x + 3)$ is

(A) $\left[\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right]$ (B) $\left[\dfrac{\pi}{4}, \pi\right)$ (C) $(0, \pi)$ (D) $\left(0, \dfrac{3\pi}{4}\right]$

6. The greatest value of the function, $f(x) = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is:

(A) $\dfrac{\pi^3}{32}$ (B) $\dfrac{\pi^3}{8}$ (C) $\dfrac{3\pi^3}{8}$ (D) $\dfrac{7\pi^3}{8}$

7. Let $f(x) = \dfrac{x - [x]}{1 + x - [x]}$, $x \in R$. Then range of f (x) = ? (Here [.] denotes greatest integer function)

(A) $\left(0, \dfrac{1}{2}\right]$ (B) $\left[0, \dfrac{1}{2}\right]$ (C) $\left[0, \dfrac{1}{2}\right)$ (D) $\left(0, \dfrac{1}{2}\right)$

8. The range of the functions $f(x) = \log_{\sqrt{2}}\left(2 - \log_2\left(16\sin^2 x + 1\right)\right)$ is

(A) $(-\infty, 1)$ (B) $(-\infty, 2)$ (C) $(-\infty, 1]$ (D) $(-\infty, 2]$

9. If domain of f(x) is $(-\infty, 0]$ then domain of $f(6\{x\}^2 - 5\{x\} + 1)$ is (where $\{\cdot\}$ represents fractional part function).

(A) $\bigcup_{n\in I}\left[n+\frac{1}{3}\ \ n+\frac{1}{2}\right]$ (B) $(-\infty, 0)$ (C) $\bigcup_{n\in I}\left[n+\frac{1}{6}\ \ n+1\right]$ (D) none of these

10. If $f(x) = \cot^{-1}x$ $: R^+ \to \left(0, \frac{\pi}{2}\right)$

and $g(x) = 2x - x^2$ $: R \to R$. Then the range of the function f(g(x)) wherever define is

(A) $\left(0, \frac{\pi}{2}\right)$ (B) $\left(0, \frac{\pi}{4}\right)$ (C) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right)$ (D) $\left(\frac{\pi}{4}\right)$

11. If $f(x) = \frac{4a-7}{3}x^3 + (a-3)x^2 + x + 5$ is a one-one function, then

(A) $2 \le a \le 8$ (B) $1 \le a \le 2$ (C) $0 \le a \le 1$ (D) None of these

12. Let f: $(e, \infty) \to R$ be defined by $f(x) = \ell n(\ell n(\ell n\, x))$, then

(A) f is one one but not onto
(B) f is on to but not one - one
(C) f is one-one and onto
(D) f is neither one-one nor onto

13. If f (x) = 2 [x] + cos x, then f: $R \to R$ is: (where [ ] denotes greatest integer function)

(A) one–one and onto
(B) one–one and into
(C) many–one and into
(D) many–one and onto

14. The function $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is

(A) an odd function
(B) an even function
(C) neither an odd nor an even function
(D) a periodic function

15. If the graph of the function $f(x) = \frac{a^x - 1}{x^n(a^x + 1)}$ is symmetric about y-axis, then n is equal to:

(A) 2 (B) 2 / 3 (C) 1 / 4 (D) – 1 / 3

16. The period of $\sin\frac{\pi}{4}[x] + \cos\frac{\pi x}{2} + \cos\frac{\pi}{3}[x]$, where [x] denotes the integral part of x is.

(A) 8 (B) 12 (C) 24 (D) Non-periodic

17. The fundamental period of function $f(x) = [x] + \left[x+\frac{1}{3}\right] + \left[x+\frac{2}{3}\right] - 3x + 15$

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) 1 (D) non-periodic

18. f (x) = $|x - 1|$, f: $R^+ \to R$

g (x) = e^x, g: $[-1, \infty) \to R$ If the function fog (x) is defined, then its domain and range respectively are:

(A) $(0, \infty)$ & $[0, \infty)$
(B) $[-1, \infty)$ & $[0, \infty)$
(C) $[-1, \infty)$ & $\left[1-\frac{1}{e}, \infty\right)$
(D) $[-1, \infty)$ & $\left[\frac{1}{e}-1, \infty\right)$

19. Let f (2, 4) $\to$ (1, 3) be a function defined by $f(x) = x - \left[\frac{x}{2}\right]$ (where [.] denotes the greatest integer function), then $f^{-1}(x)$ is equal to :

(A) 2x (B) $x + \left[\frac{x}{2}\right]$ (C) x + 1 (D) x – 1

More than one correct option's

20. For the function $f(x) = \ell n \quad (\sin^{-1} \ell og_2 x)$,

(A) Domain is $\left[\frac{1}{2}, 2\right]$ (B) Range is $\left(-\infty, \ell n\frac{\pi}{2}\right]$

(C) Domain is (1, 2] (D) Range is R

21. The mapping $f : R \to R$ given by $f(x) = x^3 + ax^2 + bx + c$ is a bijection if

(A) $b^2 \le 3a$ (B) $a^2 \le 3b$ (C) $a^2 \ge 3b$ (D) $b^2 \ge 3a$

22. In the following functions defined from [–1, 1] to [–1, 1] the fucntions which are not bijective are

(A) $\sin(\sin^{-1}x)$ (B) $\frac{2}{\pi}\sin^{-1}(\sin x)$ (C) $(\text{sgn } x)\, \ell n\, e^x$ (D) $x^3 \text{ sgn } x$

23. Function $f(x) = \sin x + \tan x + \text{sgn}(x^2 - 6x + 10)$ is

(A) periodic with period 2π (B) periodic with period π

(C) Non-periodic (D) periodic with period 4π

24. A function 'f' from the set of natural numbers to integers defined by,

$$f(n) = \begin{cases} \frac{n-1}{2} & , \text{ when n is odd} \\ -\frac{n}{2} & , \text{ when n is even} \end{cases} \text{ is:}$$

(A) one-one (B) many-one (C) onto (D) into

PART - II : SUBJECTIVE QUESTIONS

1. Check whether $x^2 + y^2 = 36$ represents a function or not.

2. If $f(x) = \frac{4^x}{4^x + 2}$, then show that $f(x) + f(1 - x) = 1$

3. Find the domain of each of the following functions :

(i) $f(x) = \frac{x^3 - 5x + 3}{x^2 - 1}$ (ii) $f(x) = \frac{\sin^{-1} x}{x}$

(iii) $f(x) = \frac{1}{\sqrt{x + |x|}}$ (iv) $f(x) = \sqrt{1 - 2x} + 3\sin^{-1}\left(\frac{3x - 1}{2}\right)$

(v) $f(x) = 2^{\sin^{-1} x} + \frac{1}{\sqrt{x - 2}}$ (vi) $\log_x \log_2\left(\frac{1}{x - 1/2}\right)$

(vii) $f(x) = \sqrt{3 - 2^x - 2^{1-x}}$ (viii) $f(x) = \sqrt{\frac{x-2}{x+2}} + \sqrt{\frac{1-x}{1+x}}$

(ix) $f(x) = \sqrt{\tan x - \tan^2 x}$ (x) $f(x) = \ell og_{10}(1 - \ell og_{10}(x^2 - 5x + 16))$

4. Draw the graph of each of the following functions given by

(i) f(x) = sgn ([x]) where [x] denotes greatest integer function

(ii) $f(x) = \begin{cases} 1, & \text{if } x \le 0 \\ x^2+1, & \text{if } 0 < x < 2 \\ 5, & \text{if } x \ge 2 \end{cases}$ (iii) $f(x) = e^{\{x\}}$ (iv) $y = [x] + \sqrt{\{x\}}$

5. Find the range of each of the following functions :

(i) $f(x) = \dfrac{x}{1+x^2}$ (ii) $f(x) = \sqrt{16-x^2}$ (iii) $f(x) = \dfrac{|x-4|}{x-4}$

(iv) $f(x) = \dfrac{1}{\sqrt{x-5}}$ (v) $f(x) = \dfrac{1}{2-\cos 3x}$ (vi) $f(x) = 3\sin\sqrt{\dfrac{\pi^2}{16}-x^2}$

(vii) $f(x) = x^3 - 12x$, where $x \in [-3, 1]$ (viii) $f(x) = \sin^2 x + \cos^4 x$

(ix) $f(x) = 5 + 3\sin x + 4\cos x$ (x) $f(x) = \ell n(\sin^{-1}x)$ (xi) $3|\sin x| - 4|\cos x|$

6. Find the domain and the range of each of the following functions :

(i) $f(x) = \dfrac{1}{\sqrt{4+3\sin x}}$ (ii) f(x) = x ! (iii) $f(x) = \dfrac{x^2-9}{x-3}$ (iv) $f(x) = \sin^2(x^3) + \cos^2(x^3)$

7. Check whether following pairs of functions are identical or not ?

(i) $f(x) = \sqrt{x^2}$ & $g(x) = \left(\sqrt{x}\right)^2$ (ii) $f(x) = \sec(\sec^{-1}x)$ & $g(x) = \text{cosec}(\text{cosec}^{-1}x)$

(iii) $f(x) = \sqrt{\dfrac{1+\cos 2x}{2}}$ & $g(x) = \cos x$ (iv) f(x) = x and $g(x) = e^{\ell nx}$

(v) $f(x) = \sin^{-1}x + \cos^{-1}x$ and $g(x) = \dfrac{\pi}{2}$ (vi) $f(x) = \tan^{-1}x + \cot^{-1}x$ and $g(x) = \dfrac{\pi}{2}$

(vii) $f(x) = e^{\ell n \sec^{-1} x}$ & $g(x) = \sec^{-1}x$ (viii) $f(x) = \cot^2 x.\cos^2 x$ & $g(x) = \cot^2 x - \cos^2 x$

(ix) $\sqrt{1+\sin x}$, $\sin\dfrac{x}{2} + \cos\dfrac{x}{2}$ (x) $\ell n\, x^3 + \ell n\, x^2$, $5\,\ell n\, x$

(xi) $e^{(\ell nx)/2}$ and $\sqrt{x}$ (xii) $\cos^2 x + \sin^4 x$ and $\sin^2 x + \cos^4 x$

8. Find for what values of x, the following functions would be identical.

$f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\dfrac{x-1}{x-2}\right)$.

9. Let $f(x) = x^2$, $g(x) = \sin x$, $h(x) = \sqrt{x}$, then verify that [fo (goh)] (x) & [(fog) oh] (x) are equal.

10. Find fog and gof, if

(i) $f(x) = e^x$; $g(x) = \log x$ (ii) $f(x) = |x|$; $g(x) = \sin x$

(iii) $f(x) = \sin^{-1}x$; $g(x) = x^2$ (iv) $f(x) = x^2 + 2$; $g(x) = 1 - \dfrac{1}{1-x}$, $x \ne 1$

11. If $f(x) = \ell n(x^2 - x + 2)$; $R^+ \to R$ and

g(x) = {x} + 1 ; [1, 2] → [1, 2] , where {x} denotes fractional part of x.

Find the domain and range of f(g(x)) when defined.

12. Let $f(x) = \begin{cases} 1+x & , \ 0 \le x \le 2 \\ 3-x & , \ 2 < x \le 3 \end{cases}$. Find fof.

13. Find whether the following functions are one-one or many-one

(i) $f(x) = |x^2 + 5x + 6|$

(ii) $f(x) = |\log x|$

(iii) $f(x) = \sin 4x, x \in \left(-\frac{\pi}{8}, \frac{\pi}{8}\right)$

(iv) $f(x) = x + \frac{1}{x}, x \in (0, \infty)$

(v) $f(x) = \sqrt{1-e^{\left(\frac{1}{x}-1\right)}}$

(vi) $f(x) = \frac{3x^2}{4\pi} - \cos \pi x$

(vii) $f(x) = \sin^{-1} x - \cos^{-1} x$

14. Let $f : A \to A$ where $A = \{x : -1 \le x \le 1\}$. Find whether the following function are bijective.

(i) $x - \sin x$ (ii) $x|x|$ (iii) $\tan \frac{\pi x}{4}$ (iv) x^4

15. Let $f : D \to R$ where D is its domain. Find whether the following functions are into/onto.

(i) $f(x) = \frac{1+x^6}{x^3}$ (ii) $f(x) = x \cos x$ (iii) $f(x) = \frac{1}{\sin\sqrt{|x|}}$ (iv) $\tan(2 \sin x)$

16. Classify the following functions f(x) defined in $R \to R$ as injective, surjective, both or none.

(i) $f(x) = x|x|$

(ii) $f(x) = x^2$

(iii) $f(x) = \frac{x^2}{1+x^2}$

(iv) $f(x) = x^3 - 6x^2 + 11x - 6$

17. Determine whether the following functions are even or odd or neither even nor odd :

(i) $\sin(x^2 + 1)$ (ii) $x + x^2$ (iii) $x - x^3$ (iv) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$

(v) $f(x) = \log(x + \sqrt{x^2+1})$ (vi) $f(x) = \sin x + \cos x$ (vii) $f(x) = \frac{(1+2^x)^7}{2^x}$

(viii) $f(x) = \frac{\sec x + x^2 - 9}{x \sin x}$ (ix) $f(x) = \frac{2x(\sin x + \tan x)}{2\left[\frac{x+2\pi}{\pi}\right] - 3}$, where [ ] denotes greatest integer function.

18. Find the fundamental period of the following function

(i) $f(x) = 2 + 3\cos(x - 2)$

(ii) $f(x) = \sin\frac{\pi x}{4} + \sin\frac{\pi x}{3}$

(iii) $f(x) = [\sin 3x] + |\cos 6x|$

(iv) $f(x) = \frac{\sin 12x}{1+\cos^2 6x}$

(v) $f(x) = \tan\frac{\pi}{2}[x]$, where [.] denotes greatest integer function.

(vi) $f(x) = e^{\ln \sin x} + \tan^3 x - \operatorname{cosec}(3x - 5)$

(vii) $f(x) = \sin x + \tan\frac{x}{2} + \sin\frac{x}{2^2} + \tan\frac{x}{2^3} + \ldots\ldots\ldots + \sin\frac{x}{2^{n-1}} + \tan\frac{x}{2^n}$

19. Find the period of f(x) satisfying the condition :

(i) $f(x + p) = 1 + \{1 - 3f(x) + 3f^2(x) - f^3(x)\}^{1/3}$

(ii) $f(x - 1) + f(x + 3) = f(x + 1) + f(x + 5)$

20. Let $f : D \to R$, where D is the domain of f. Find the inverse of f, if it exists, where

(i) $f(x) = 1 - 2^{-x}$

(ii) $f(x) = \left(4-(x-7)^3\right)^{1/5}$

(iii) $f(x) = \ell n\,(x + \sqrt{1+x^2})$

(iv) $f(x) = \frac{e^{2x} - e^{-2x}}{2}$

21. Let $f : \left[-\frac{\pi}{3}, \frac{\pi}{6}\right] \to B$ defined by $f(x) = 2\cos^2 x + \sqrt{3}\sin 2x + 1$. Find B such that f^{-1} exists. Also find $f^{-1}(x)$.

22. Solve $2x^2 - 5x + 2 = \frac{5-\sqrt{9+8x}}{4}$ where $x < \frac{5}{4}$

23. Let f(x) be a polynomial function satisfying the relation $f(x).\,f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)\ \forall\ x \in R - \{0\}$ and $f(3) = -26$. Determine $f'(1)$.

24. If $f(x + y) = f(x).\,f(y)\ \forall\ x, y \in N$ and $f(1) = 2$, then find $\sum_{n=1}^{10} f(n)$.

25. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$ where the function 'f' satisfies the relation $f(x + y) = f(x).\,f(y)$ for all natural numbers x, y and further $f(1) = 2$

26. Let f(x) be defined on [–2, 2] and is given by $f(x) = \begin{cases} -1 & , \ -2 \le x < 0 \\ x-1 & , \ 0 \le x \le 2 \end{cases}$ and $g(x) = f(|x|) + |f(x)|$, then find g(x).

EXERCISE # 3

PART-I IIT-JEE (PREVIOUS YEARS PROBLEMS)

*** Marked Questions are having more than one correct option.**

1. Suppose $f(x) = (x + 1)^2$ for $x \ge -1$. If g(x) is the function whose graph is the reflection of the graph of f(x) with respect to the line y = x, then g(x) equals : **[IIT-2002]**

(A) $-\sqrt{x} - 1,\ x \ge 0$ (B) $\frac{1}{(x+1)^2},\ x > -1$ (C) $\sqrt{x+1},\ x \ge -1$ (D) $\sqrt{x} - 1,\ x \ge 0$

2. Let function $f : R \to R$ be defined by $f(x) = 2x + \sin x$ for $x \in R$, then f is : **[IIT-2002]**

(A) one-to-one and onto (B) one to one but not onto

(C) onto but NOT one-to-one (D) neither one-to-one nor onto

3. If $f : [0, \infty) \to [0, \infty)$ and $f(x) = \frac{x}{1+x}$, then f is: **[IIT – 2003, 3]**

(A) one-one and onto (B) one-one but not onto

(C) onto but not one-one (D) neither one-one nor onto

4. Range of the function $f(x) = \frac{x^2+x+2}{x^2+x+1}$; $x \in R$ is : **[IIT – 2003, 3]**

(A) $(1, \infty)$ (B) $\left(1, \frac{11}{7}\right]$ (C) $\left(1, \frac{7}{3}\right]$ (D) $\left(1, \frac{7}{5}\right]$

5. Domain of definition of the function $f(x) = \sqrt{\sin^{-1}(2x) + \frac{\pi}{6}}$ for real valued 'x' is: **[IIT – 2003, 3]**

(A) $\left[-\frac{1}{4}, \frac{1}{2}\right]$ (B) $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (C) $\left(-\frac{1}{2}, \frac{1}{9}\right)$ (D) $\left[-\frac{1}{4}, \frac{1}{4}\right]$

6. If $f(x) = x^2 - 1$ and $g(x) = \sin x + \cos x$, then f(g(x)) is invertible in which of the following interval? **[JEE 2004]**

(A) $\left[0, \frac{\pi}{2}\right]$ (B) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (C) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ (D) $[0, \pi]$

7. If X and Y are two sets and $f : X \to Y$. If $\{f(c) = y ;\ c \subset X,\ y \subset Y\}$ and $\{f^{-1}(d) = x ;\ d \subset Y,\ x \subset X\}$, then the true statement is : **[JEE 2005 (Screening),3]**

(A) $f(f^{-1}(b)) = b$ (B) $f(f^{-1}(a)) = a$

(C) $f(f^{-1}(b)) = b,\ b \not\subset y$ (D) $f^{-1}(f(a)) = a,\ a \subset x$

8. If function f (x) and g (x) are defined on $R \to R$ such that

$f(x) = \begin{cases} 0, & x \in \text{rational} \\ x, & x \in \text{irrational} \end{cases}$, $g(x) = \begin{cases} 0, & x \in \text{irrational} \\ x, & x \in \text{rational} \end{cases}$, then (f – g) (x) is : **[JEE 2005 (Screening),3]**

(A) one-one and onto (C) neither one-one nor onto

(C) one-one not onto (D) onto not one-one

9. **Match the column**

Let $f(x)=\dfrac{x^2-6x+5}{x^2-5x+6}$ **[JEE 2007]**

Match the expressions/statements in Column I with expressions/statements in Column II

Column- I		**Column-II**	
(A)	If $-1 < x < 1$, then f(x) satisfies	(p)	$0 < f(x) < 1$
(B)	If $1 < x < 2$, then f(x) satisfies	(q)	$f(x) < 0$
(C)	If $3 < x < 5$, then f(x) satisfies	(r)	$f(x) > 0$
(D)	If $x > 5$, then f(x) satisfies	(s)	$f(x) < 1$

10. The maximum value of function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x \mid x^2 + 20 \le 9x\}$ is

[JEE 2009]

11. If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[JEE 2009]**

12. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(fog)(x) = f(g(x))$, is : **[JEE 2011]**

(A) $\pm\sqrt{n\pi}$, $n \in \{0, 1, 2, ...\}$

(B) $\pm\sqrt{n\pi}$, $n \in \{1, 2, ...\}$

(C) $\frac{\pi}{2} + 2\pi$, $n \in \{.... -2, -1, 0, 1, 2, ...\}$

(D) $2n\pi$, $n \in \{..... -2, -1, 0, 1, 2\}$

13. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is : **[JEE 2012]**

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

14. Let $f : (-1, 1) \to IR$ be such that $f(\cos 4\theta) = \dfrac{2}{2-\sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the value(s) of $f\left(\frac{1}{3}\right)$ is : (are) **[JEE 2012]**

(A) $1-\sqrt{\frac{3}{2}}$

(B) $1+\sqrt{\frac{3}{2}}$

(C) $1-\sqrt{\frac{2}{3}}$

(D) $1+\sqrt{\frac{2}{3}}$

PART-II AIEEE (PREVIOUS YEARS PROBLEMS)

1. A function f from the set of natural numbers to intergers defined by $f(n) = \begin{cases} \frac{n-1}{2}, \text{where n is odd} \\ -\frac{n}{2}, \text{when n is even} \end{cases}$ is :

(1) one-one but not onto (2) onto but not one-one. **[AIEEE 2003]**

(3) one-one and onto both. (4) neither one-one nor onto.

2. If f : R $\rightarrow$ R satisfies f(x + y) = f(x) + f(y), for all x, y $\in$ R and f(1) = 7, then $\sum_{r=1}^{n} f(r)$ is : **[AIEEE 2003]**

(1) $\frac{7n}{2}$ (2) $\frac{7(n+1)}{2}$ (3) 7n (n +1) (4) $\frac{7n(n+1)}{2}$.

3. The range of the function f(x) = $^{7-x}P_{x-3}$ is : **[AIEEE 2004]**

(1) {1,2,3} (2) {1,2,3,4,5,6} (3) {1,2,3,4} (4) {1,2,3,4,5}

4. A real valued function f(x) satisfies the functional equation f(x – y) = f(x) f(y) – f(a – x) f(a + y), where a is a given constant and f(0) = 1, then f(2a – x) is equal to : **[AIEEE 2005]**

(1) f(–x) (2) f(1) + f(a – x) (3) f(x) (4) – f(x)

5. The largest interval lying in $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ for which the function f(x) = $4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right)$ + log (cos x) is defined, is : **[AIEEE 2007]**

(1) [0, π] (2) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (3) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (4) $\left[0, \frac{\pi}{2}\right)$

6. Let f : N $\rightarrow$ Y be a function defined as f(x) = 4x + 3 where

Y = {y $\in$ N : y = 4x + 3 for some x $\in$ N}

Show that f is invertible and its inverse is : **[AIEEE 2008]**

(1) $g(y) = \frac{y-3}{4}$ (2) $g(y) = \frac{3y+4}{3}$ (3) $g(y) = 4 + \frac{y+3}{4}$ (4) $g(y) = \frac{y+3}{4}$

7. For real x let f(x) = x^3 + 5x + 1, then : **[AIEEE 2009]**

(1) f is one-one but not onto R (2) f is onto R but not one-one

(3) f is one one and onto R (4) f is neither one-one nor onto R

8. Let f(x) = $(x + 1)^2 - 1$, x $\geq$ – 1. **[AIEEE 2009]**

Statement I The set {x : f(x) = f^{-1}(x)} = {0, – 1}

Statement II f is a bijection

(1) Statement I is true, Statement II is true; Statement II is a correct explanation for statement I.

(2) Statement I is true, Statement II is true ; Statement II is not a correct explanation for Statement I.

(3) Statement I is true, Statement II is false.

(4) Statement I is false, Statement II is true.

9. The domain of the function f(x) $\frac{1}{\sqrt{|x|-x}}$ is : **[AIEEE 2011]**

(1) $(0, \infty)$ (2) $(-\infty, 0)$ (3) $(-\infty, \infty) - \{0\}$ (4) $(-\infty, \infty)$

10. Let X = {1, 2, 3, 4, 5}. The number of different ordered pairs (Y, Z) that can be formed such that $Y \subseteq X$, $Z \subseteq X$, and $Y \cap Z$ is empty, is : **[AIEEE 2012]**

(1) 5^2 (2) 3^5 (3) 2^5 (4) 5^3

EXERCISE # 4

NCERT BOARD QUESTIONS

1. If $f(x) = 3x^4 - 5x^2 + 9$, find $f(x-1)$

2. If $f(x) = x + \frac{1}{x}$, prove that $[f(x)]^3 = f(x^3) + 3f\left(\frac{1}{x}\right)$

3. Let $f:[2, \infty) \to R$ and $g: [-2, \infty) \to R$ be two real functions defined by $f(x) = \sqrt{x-2}$ and $g(x) = \sqrt{x+2}$ find f + g and f– g.

4. Let f be the exponential function and g be the logarithmic function, find

(i) (f + g) (1) (ii) (fg) (1) (iii) (3f) (1) (iv) (5g) (1)

5. Find the domain of each of the following function given by

(i) $f(x) = \frac{x}{x^2 - 3x + 2}$ (ii) f(x) $\frac{1}{\sqrt{x+|x|}}$

6. Find the domain and range of each of the following function given by

(i) $f(x) = 1 - |x - 3|$ (ii) $f(x) = \frac{1}{\sqrt{4 + 3\sin x}}$ (iii) $f(x) = 1 + 3\cos 2x$

7. If $f: R \to R$ and $g: R \to R$ be functions defined by $f(x) = x^2 + 1$ and $g(x) = \sin x$, then find fog and gof.

8. If $f(x) = e^x$ and $g(x) = \log_e x$ $(x > 0)$, find fog and gof. If fog = gof?

9. If $f(x) = \sqrt{x}$ $(x \ge 0)$ and $g(x) = x^2 - 1$ are two real functions, find fog and gof. Is fog = gof ?

10. If $f(x) = \frac{1}{2x+1}$, $x \ne -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$, provided that $x \ne -\frac{1}{2}, -\frac{3}{2}$

11. Let $f(x) = \frac{x}{\sqrt{1+x^2}}$, then, show that $(fofof)(x) = \frac{x}{\sqrt{1+3x^2}}$

12. Let f be any real function and let g be a function given by g(x) = 2x. Prove that gof = f + f.

13. Let f, g, h be real functions given by f(x) = sin x, g (x) = 2x and h (x) = cos x. Prove that fog = go (fh).

14. Find fog and gof, if
(i) f (x) = x + 1, g (x) = e^x (ii) f (x) = $\sin^{-1} x$, g (x) = x^2

15. Prove that the function f: R $\to$ R , given by f(x) = 2x, is one-one and onto.

16. Show that an onto function f: {1, 2, 3} $\to$ {1, 2, 3} is always one-one.

17. Prove that the Greatest integer function f: R $\to$ R, given by f(x) = [x], is neither one-one nor onto, where [x] denotes the greatest integer less than or equal to x.

18. Show that the Modulus Function f : R $\to$ R, given by f(x) = |x|, is neither one- one nor onto, where |x| is x, if x is positive or 0 and |x| is - x, if x is negative.

19. Show that the Signum Function f: R $\to$ R, given by $f(x) = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$ is neither one-one nor onto.

20. Let f : N $\to$ N be defined by f(n) = $\begin{cases} \dfrac{n+1}{2}, \text{if n is odd} \\ \dfrac{n}{2}, \text{if n is even} \end{cases}$ for all n $\in$ N. State whether the function f is bijective . Justify your answer.

21. Let A = R – {3} and B = R – {1}. Consider the function f : A $\to$ B defined by $f(x) = \left(\dfrac{x-2}{x-3}\right)$. Is f one-one and onto? Justify your answer.

22. Give examples of two function f : N $\to$ Z and g : Z $\to$ Z such that gof is injective but g is not injective.

23. Give examples of two function f : N $\to$ N and g : N $\to$ N such that gof is onto but f is not onto.

24. Prove that the function f : R $\to$ R defined as f(x) = 2x – 3 is invertible. Also find f^{-1} .

25. Show that be function f: R $\to$ R is given by f(x) = x^2 + 1 is not invertible.

26. If f (x) = $\dfrac{x-1}{x+1}$, x $\neq$ 1, – 1, show that fof^{-1} is the identity function.

27. Find the value of parameter α for which the function f (x) = 1 + α x , $\alpha \neq 0$ is the inverse of itself.

28. Let A = {– 1, 0 , 1, 2}, B = {– 4, – 2, 0, 2} and f, g: A $\to$ B be function define by f(x) = x^2 – x , x $\in$ A and g (x) = 2 $\left|x - \dfrac{1}{2}\right|$ – 1, x $\in$ A , Are f and g equal justify your answer. (Hint: one may not that two functions f: A $\to$ B and g : A $\to$ B such that f(a) = g(a) $\forall$ a $\in$ A, are called equal functions).

ANSWERS

EXERCISE # 1

PART - I

A-1. (C) **A-2.** (D) **A-3.** (B) **A-4.** (B) **A-5.** (B) **A-6.** (A) **A-7.** (D)

A-8. (A) **A-9.** (C) **A-10.** (B) **A-11.** (A) **A-12.** (B) **A-13.** (C) **B-1.** (A)

B-2. (B) **B-3.** (D) **B-4.** (A) **B-5*.** (A, C) **B-6.** (A) **B-7*.** (B, D) **B-8*.** (A, D)

C-1. (B) **C-2.** (B) **C-3.** (C) **C-4.** (D) **C-5.** (A) **C-6*.** (A,B,C) **C-7*.** (A, B, D)

C-8*. (B, C, D) **C-9*.** (A, B, C) **D-1.** (D) **D-2.** (D) **D-3.** (D) **D-4.** (C)

PART - II

1. (A) **2.** (B) **3.** (A) 4. **(C)** 5. **(A)** 6. **(D)**

7. (A- s, B - p, C - r , D - q) **8.** (A- q , B- q , C- pqs, D- rs) **9.** (A) **10.** (A)

11. (D) **12.** (B)

EXERCISE # 2

PART - I

1. (C) **2.** (C) **3.** (C) **4.** (C) **5.** (D) **6.** (D) **7.** (C)

8. (D) **9.** (A) **10.** (C) **11.** (A) **12.** (C) **13.** (C) **14.** (B)

15. (D) **16.** (C) **17.** (A) **18.** (B) **19.** (C) **20.** (B,C) **21.** (B)

22. (B, C, D) **23.** (A, D) **24.** (A, C)

PART - II

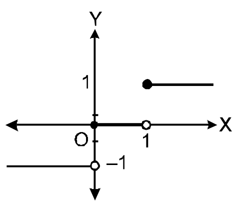
1. (No)

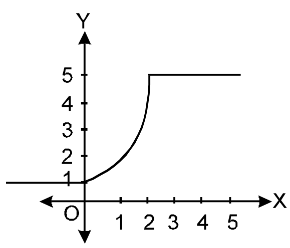
3. (i) $R - \{-1, 1\}$ (ii) $[-1, 1] - \{0\}$ (iii) $(0, \infty)$ (iv) $\left[-\frac{1}{3}, \frac{1}{2}\right]$

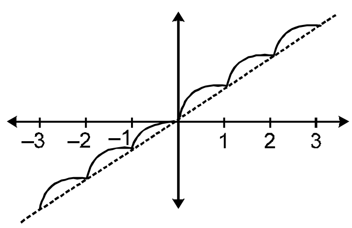
(v) ϕ (vi) $\left(\frac{1}{2}, 1\right) \cup \left(1, \frac{3}{2}\right)$ (vii) $[0, 1]$ (viii) ϕ

(ix) $\bigcup_{n \in I} \left[n\pi, n\pi + \frac{\pi}{4}\right]$

(x) $(2, 3)$

4. (i) 

(ii) 

(iv) 

5. (i) $\left[-\frac{1}{2},\frac{1}{2}\right]$ (ii) [0, 4] (iii) {–1, 1} (iv) R^+ (v) $\left[\frac{1}{3},1\right]$

(vi) $\left[0,\frac{3}{\sqrt{2}}\right]$ (vii) [–11, 16] (viii) $\left[\frac{3}{4},1\right]$ (ix) [0, 10] (x) $(-\infty, \ell n\, \pi/2]$

(xi) [–4, 3]

6. (i) Domain : R, Range : $\frac{1}{\sqrt{7}} \le y \le 1$ (ii) Domain : $N \cup \{0\}$, Range : {n! : n = 0, 1, 2,...}
(iii) Domain R – {3}, Range : R – {6} (iv) Domain : R, Range : {1}

7. (i) No (ii) Yes (iii) No (iv) No (v) No (vi) Yes
(vii) No (viii) Yes (ix) No (x) Yes (xi) No (xii) Yes

8. $(2, \infty)$ **9.** $[fo(goh)](x) = [(fog)oh](x) = \sin^2\sqrt{x}$

10. (i) fog = x, x > 0 ; gof = x, $x \in R$ (ii) |sin x|, sin |x|

(iii) $\sin^{-1}(x^2)$, $(\sin^{-1}x)^2$ (iv) $\frac{3x^2-4x+2}{(x-1)^2}, \frac{x^2+2}{x^2+1}$

11. Domain : [1 , 2] ; Range : $[\ell n2, \ell n4)$ **12.** $(fof)(x) = \begin{cases} 2+x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 2 \\ 4-x, & 2 < x \le 3 \end{cases}$

13. (i) many-one (ii) many-one (iii) one-one (iv) many-one (v) one – one (vi) many-one
(vii) one – one

14. (i) No (ii) Yes (iii) Yes (iv) No **15.** (i) into (ii) onto (iii) into (iv) onto

16. (i) bijective (injective as well as surjective) (ii) neither injective nor surjective
(iii) neither surjective nor injective (iv) surjective but not injective

17. (i) even, (ii) neither even nor odd (iii) odd
(iv) even, (v) odd, (vi) neither even nor odd
(vii) neither even nor odd (viii) even (ix) odd

18. (i) 2π (ii) 24 (iii) $\frac{2\pi}{3}$ (iv) $\pi/6$ (v) 2 (vi) 2π (vii) $2^n\pi$

19. (i) 2p (ii) 8

20. (i) f^{-1} Does not exists (ii) $f^{-1} : R \to R$; $f^{-1} = 7 + (4 - x^5)^{1/3}$

(iii) $f^{-1} : R \to R$; $f^{-1} = \frac{e^x - e^{-x}}{2}$ (iv) $f^{-1} : R \to R$, $f^{-1}(x) = \frac{1}{2}\ell n\,(x + \sqrt{x^2+1})$

21. B = [0, 4] ; $f^{-1}(x) = \frac{1}{2}\left(\sin^{-1}\left(\frac{x-2}{2}\right) - \frac{\pi}{6}\right)$ **22.** $x = \frac{3-\sqrt{5}}{2}$

23. – 3 **24.** 2046 **25.** 3 **26.** $g(x) = \begin{cases} -x, & -2 \le x < 0 \\ 0, & 0 \le x \le 1 \\ 2(x-1), & 1 < x \le 2 \end{cases}$

EXERCISE # 3

PART - I

1. (D) **2.** (A) **3.** (B) **4.** (C) **5.** (A) **6.** (C) **7.** (D)

8. (A) **9.** Ans- (A-Prs, B -qs, C-qs, D-prs) **10.** (7) **12.** (A) **13.** (B)

14. (A, B)

PART - II

1. (3) **2.** (4) **3.** (1) **4.** (4) **5.** (4) **6.** (1) **7.** (3)

8. (3) **9.** (2) **10.** (2)

EXERCISE # 4

1. $3x^4 - 12x^3 + 13x^2 - 2x + 7$ **3.** $\sqrt{x-2} + \sqrt{x+2}$, $x \geq 2$ and $\sqrt{x-2} - \sqrt{x+2}$, $x \geq 2$

4. (i) e (ii) 0 (iii) 3e (iv) 0 **5.** (i) $R - \{1, 2\}$ (ii) $(0, \infty)$

6. (i) R, $(-\infty, 1]$ (ii) R, $\left[\frac{1}{\sqrt{7}}, 1\right]$ (iii) R, $[-2, 4]$ **7.** $\sin^2 x + 1$, $\sin(x^2+1)$

8. x, x, No **9.** $\frac{1}{\sqrt{x^2-1}}$, $x - 1$, No **14.** (i) $e^x + 1$, e^{x+1} (ii) $\sin^{-1}(x^2)$, $(\sin^{-1} x)^2$

20. No **21.** Yes **22.** $f(x) = x$, $g(x) = |x|$

23. $f(x) = x + 1$, $g(x) = \{x - 1$ if $x > 1$ and 1 if $x = 1\}$

24. $\frac{x+3}{2}$ **27.** $\alpha = -1$ **28.** Yes